Application of Biorthogonal Interpolating Wavelets to the Galerkin Scheme of Time Dependent Maxwell's Equations

Masafumi Fujii and Wolfgang J. R. Hoefer

***Abstract*—A family of biorthogonal interpolating wavelets has been applied to time-domain electromagnetic field modeling through the wavelet-Galerkin scheme. The scaling functions are the Deslauriers–Dubuc interpolating functions and the wavelets are the shifted and contracted version of the scaling functions. This set of bases yields a simple algorithm for the solution of Maxwell's equations in time domain due to their interpolation properties. The derivation of the algorithm is presented in this paper, followed by a series of numerical verification on some resonant structures.**

***Index Terms*—Biorthogonal wavelets, Deslauriers–Dubuc interpolating functions, electromagnetic field analysis, time domain.**

I. INTRODUCTION

WAVELETS have been applied to the solution of Maxwell's equations in time domain [1], [2] yielding a scheme of highly linear numerical dispersion properties. The authors have extended the shifted interpolation scheme of [2] by using the scaling functions of higher regularity [3] and have shown advantages of the scheme in analyzing electrically large inhomogeneous structures [4].

This paper addresses an application of the interpolating wavelets [5], [6] through the wavelet-Galerkin scheme of the time dependent Maxwell's equations. The scaling function is the Deslauriers–Dubuc interpolating function [7], and the wavelet is the shifted and contracted version of the scaling function. These functions constitute non-L^2 biorthogonal bases that are smooth, symmetric, compactly supported and exactly interpolating. Unlike the Daubechies orthogonal wavelets [8], of which interpolation property is limited to the bases of low regularity [3], the proposed basis set yields a scheme of an arbitrary order of regularity as well as saves the computational overhead of total filed reconstruction.

II. FORMULATION

The Deslauriers–Dubuc interpolating function ϕ of order $2p-1$ is given by an autocorrelation function of the Daubechies compactly supported orthogonal scaling functions ϕ_0 of p vanishing moments [9] as

$$\phi(x) = \int_{-\infty}^{+\infty} \phi_0(u)\phi_0(u-x)\,du \tag{1}$$

which has even symmetry and minimum support of $[-2p+1, 2p-1]$ to reproduce polynomials of order $2p-1$. Fig. 1 shows $\phi(x)$ of $p = 2, \cdots, 10$.

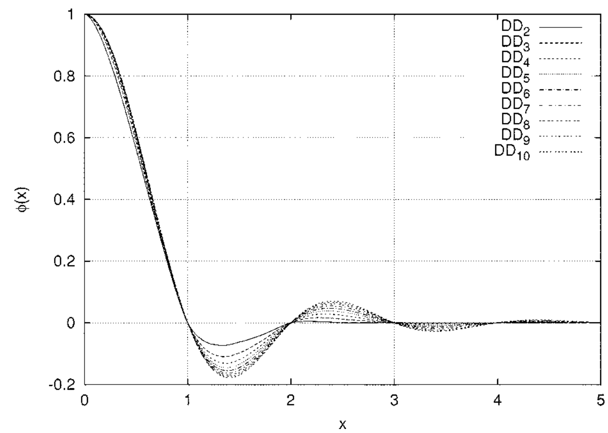

Fig. 1. Deslauriers–Dubuc interpolating function of order $2p-1$ for $p = 2, \cdots, 10$. They have even symmetry with respect to $x = 0$. The numbers in the legend denote the parameter p.

We choose $\phi(x)$ as a scaling function, which satisfies the so-called dilation relation

$$\phi(x) = \sum_{k=-\infty}^{+\infty} h_k^* \phi(2t-k). \tag{2}$$

The filter coefficients h_k^* in (2) are obtained from the Daubechies filter of the compactly supported wavelets h_k [8] by

$$h_k^* = \sum_{m=-\infty}^{+\infty} h_m h_{m-k}. \tag{3}$$

For $p=2, \{h_k^* | k=-3, -2, \cdots, 3\} = \{-0.0625, 0, 0.5625, 1.0, 0.5625, 0, -0.0625\}$ and for $p=4$, $\{h_k^* | k = -7, -6, \cdots, 7\} = \{-0.002\,441\,41, 0, 0.023\,925\,8, 0, -0.119\,629, 0, 0.598145, 1.0, 0.598145, 0, -0.119\,629, 0, 0.023\,925\,8, 0, -0.002\,441\,41\}$. Then, the wavelet function which creates the "detail" space can be chosen as

$$\psi(x) = \phi(2x-1). \tag{4}$$

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TABLE I
CONNECTION COEFFICIENTS FOR THE INTERPOLATING BASIS OF $p = 4$

l	$a_{\phi\phi}$	$a_{\phi\psi}$	$a_{\psi\phi}$	$a_{\psi\psi}$
-8		-0.0000018		
-7		0.0000397		
-6	-0.0000109	-0.0004033		0.0000109
-5	-0.0008309	0.0030967		0.0008309
-4	0.0086543	-0.0144141	-0.0000017	-0.0086543
-3	-0.0419957	0.0394952	0.0044481	0.0426846
-2	0.1560101	-0.0637289	0.3839979	-0.2903309
-1	-1.3110341	0.0533016	0.0	-1.8610039
0	1.3110341	0.0	-0.3839979	1.8610039
1	-0.1560101	-0.0533016	-0.0044481	0.2903309
2	0.0419957	0.0637289	0.0000017	-0.0426846
3	-0.0086543	-0.0394952		0.0086543
4	0.0008309	0.0144141		-0.0008309
5	0.0000109	-0.0030967		-0.0000109
6		0.0004033		
7		-0.0000397		
8		0.0000018		

Although (4) is not a true wavelet since it has no vanishing moments, it generates multiresolution analysis and plays the same role as other wavelets in the context of the Galerkin procedure.

The dual functions can be chosen as the Dirac delta function and a linear combination of those as

$$\tilde{\phi}(x) = \delta(x) \tag{5}$$

$$\tilde{\psi}(x) = \sum_{k=-2p+2}^{2p} (-1)^{k-1} h^*_{-k+1} \delta\left(x - \frac{k}{2}\right). \tag{6}$$

Since the Dirac delta is not in the space of square integrable functions, the resulting basis set is in non-L^2 space. Let $f_j(x) = f((x/\Delta x) - j)$ for $f = \phi, \psi, \tilde{\phi}$, and $\tilde{\psi}$ with Δx being the spatial discretization interval, then the set of the basis functions satisfies the biorthogonal relations

$$\begin{aligned} &\langle \phi_i, \tilde{\phi}_j \rangle = \delta_{ij}, \quad \langle \psi_i, \tilde{\psi}_j \rangle = \delta_{ij}, \\ &\langle \phi_i, \tilde{\psi}_j \rangle = \langle \psi_i, \tilde{\phi}_j \rangle = 0 \end{aligned} \tag{7}$$

where δ is the Kronecker delta function.

The electromagnetic field is then expanded in the scaling functions (1) and the wavelet functions (4) in space and in the Haar scaling functions $h(t)$ [1] in time; for the case of two-dimensional (2-D) TE polarization

$$\begin{aligned} &E_y(x, z, t) \\ &= \sum_{i,k,n=-\infty}^{+\infty} \{E^{y,\phi\phi}_{i,k,n+1/2} \phi_i(x)\phi_k(z) \\ &+ E^{y,\phi\psi}_{i,k+1/2,n+1/2} \phi_i(x)\psi_k(z) + E^{y,\psi\phi}_{i+1/2,k,n+1/2} \psi_i(x)\phi_k(z) \\ &+ E^{y,\psi\psi}_{i+1/2,k+1/2,n+1/2} \psi_i(x)\psi_k(z)\} h_{n+1/2}(t) \end{aligned} \tag{8}$$

and similarly for H_x and H_z, where $h_n(t) = h(t/\Delta t - n + 1/2)$ and Δt being the time step. Note that, as in [1], expansion coefficients for the wavelet terms are defined on the Yee cell [10] at nodes halfway between the regular nodes in the direction of the corresponding wavelets.

The field expansions (8) and those of H_x and H_z are then substituted into Maxwell's curl equations

$$-\mu \frac{\partial H_x}{\partial t} = -\frac{\partial E_y}{\partial z}, \quad -\mu \frac{\partial H_z}{\partial t} = \frac{\partial E_y}{\partial x}$$

$$J_y + \sigma E_y + \epsilon \frac{\partial E_y}{\partial t} = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}. \tag{9}$$

For the Galerkin procedure of biorthogonal bases, the equation is tested with the dual functions (5) and (6), and by using the biorthogonality conditions (7) we obtain the final time-evolution equations in the similar form as in the W-MRTD scheme [1]

$$\begin{aligned} &E^{y,\phi\phi}_{i,k,n+1/2} \\ &= \frac{2\epsilon - \sigma\Delta t}{2\epsilon + \sigma\Delta t} E^{y,\phi\phi}_{i,k,n-1/2} + \frac{2\Delta t}{2\epsilon + \sigma\Delta t} \\ &\cdot \left\{ \frac{1}{\Delta z} \left[\sum_l a_{\phi\phi}(l) H^{x,\phi\phi}_{i,k+l+1/2,n} + \sum_l a_{\psi\phi}(l) H^{x,\phi\psi}_{i,k+l+1,n} \right] \right. \\ &\quad - \frac{1}{\Delta x} \left[\sum_l a_{\phi\phi}(l) H^{z,\phi\phi}_{i+l+1/2,k,n} \right. \\ &\quad \left. \left. + \sum_l a_{\psi\phi}(l) H^{z,\phi\psi}_{i+l+1/2,k+1/2,n} \right] - J^{y,\phi\phi}_{i,k,n} \right\}. \end{aligned} \tag{10}$$

The field components of H can be obtained similarly. The summation with respect to the stencil l in (10) is taken according to the number of the connection coefficients $a_{\xi\zeta}$ for $\xi, \zeta = \phi, \psi$. The dual functions in the form of the Dirac impulses simplify the calculation of the connection coefficients $a_{\xi\zeta}(l)$ as

$$a_{\phi\phi}(l) \equiv \left\langle \frac{d\phi_{i+1/2}}{dx} \middle| \tilde{\phi}_{i-l} \right\rangle = \left. \frac{d\phi(x)}{dx} \right|_{x=-l-1/2} \tag{11}$$

$$\begin{aligned} a_{\phi\psi}(l) &\equiv \left\langle \frac{d\phi_{i+1/2}}{dx} \middle| \tilde{\psi}_{i-l} \right\rangle \\ &= \sum_{k=-2p+2}^{2p} (-1)^{k-1} h^*_{-k+1} \left. \frac{d\phi(x)}{dx} \right|_{x=-l+k/2-1/2} \end{aligned} \tag{12}$$

$$a_{\psi\phi}(l) \equiv \left\langle \frac{d\psi_{i+1/2}}{dx} \middle| \tilde{\phi}_{i-l} \right\rangle = 2 \left. \frac{d\phi(x)}{dx} \right|_{x=-2l-2} \tag{13}$$

$$\begin{aligned} a_{\psi\psi}(l) &\equiv \left\langle \frac{d\psi_{i+1/2}}{dx} \middle| \tilde{\psi}_{i-l} \right\rangle \\ &= 2 \sum_{k=-2p+2}^{2p} (-1)^{k-1} h^*_{-k+1} \left. \frac{d\phi(x)}{dx} \right|_{x=-2l+k-2} \end{aligned} \tag{14}$$

which have been evaluated numerically and listed in Table I for $p = 4$ as an example.

Although the wavelet terms in (10) is valid only for a homogeneous medium, the scheme for inhomogeneous media is given as those in [1] with somewhat simpler representation. It can be shown that $a_{\phi\phi}$ and $a_{\psi\psi}$ of the present scheme are equivalent to those of the Daubechies orthogonal wavelets. This fact implies that the numerical dispersion of the scheme using the scaling function is equivalent to that in [3], and that the present scheme yields an optimum algorithm with a minimum number of connection coefficients to reproduce a particular order of polynomial.

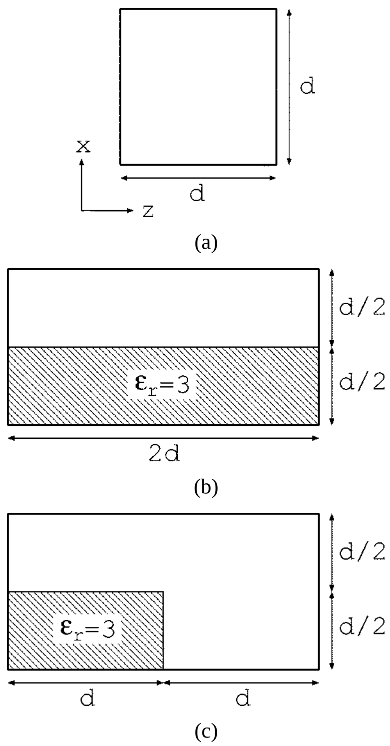

Fig. 2. Resonant structures investigated: (a) an air-filled square resonator, (b) partially dielectric filled cavity with one-dimensional material boundary, and (c) partially dielectric filled cavity with 2-D material boundary. The dimension $d = 0.5\sqrt{2}$ is a normalized value.

TABLE II
COMPUTED DOMINANT RESONANT FREQUENCY F_r, AND CPU TIME (s) OF THE RESONATORS. THE REFERENCE VALUES OF THE DOMINANT RESONANT FREQUENCIES ARE (a): 1.0, (b): 0.540 009 0, AND (c): 0.592 428 9

Resonator	No. of Yee cells and basis functions used		DD_2	DD_4	DD_{10}
a	2×2 $\phi\phi$ only	F_r	1.014994	1.012767	1.002264
		error (%)	+1.50	+1.27	+0.226
		CPU time (s)	0.38	0.40	0.42
	2×2 $\phi\phi$, $\phi\psi$ and $\psi\phi$	F_r	0.991525	1.001945	1.002047
		error (%)	-0.847	+0.195	+0.205
		CPU time (s)	0.50	0.59	0.65
b	4×8 $\phi\phi$ only	F_r	0.546939	0.546171	0.545802
		error (%)	+1.28	+1.14	+1.07
		CPU time (s)	1.56	1.94	2.01
c	4×8 $\phi\phi$ only	F_r	0.602563	0.601690	0.601281
		error (%)	+1.71	+1.56	+1.49
		CPU time (s)	1.54	1.91	2.06

III. VERIFICATION OF THE METHOD

Resonator structures depicted in Fig. 2 were analyzed with the proposed method of $p = 2, 4,$ and 10 (denoted as DD_2, DD_4 and DD_{10}). Cavity (a) was filled with air and analyzed with two schemes: 1) the scaling function only [$\phi(x)\phi(z)$] and 2) the scaling and the wavelet functions [$\phi(x)\phi(z)$, $\phi(x)\psi(z)$, $\psi(x)\phi(z)$]. Cavity (b) and (c) were inhomogeneous cases and analyzed with the scaling function only.

Square grid $\Delta l \equiv \Delta x = \Delta z$ was used in the analysis. The stability criterion is given in [11]; the stability factor $s \equiv c_0\Delta t/\Delta l$ for the present scheme was chosen to be $s = 0.1$. The dimensions are all normalized such that the speed of light is unity. The perfect electric conductor (PEC) condition was implemented using the mirror principle as in [1], imposing even symmetry for the tangential electric field and odd symmetry for the tangential magnetic field. The number of time steps were 2829 for cavity (a) and 5656 for cavities (b) and (c) to obtain a convergent frequency spectrum. The dominant resonant frequencies are listed in Table II.

It was found that, for cavity (a), DD_2 and DD_4 gave about 1% of error, and by using DD_{10}, the error decreased to about 0.2% while only a few cells are involved. Note also that by adding wavelets $\phi\psi$ and $\psi\phi$ in DD_4, the error decreased to the same level. For the case of inhomogeneous media b) and c), the numerical errors are larger than the homogeneous case a); this is due to the inaccurate representation of dielectric interface in this scheme.

IV. CONCLUSION

The biorthogonal interpolating wavelets have been applied to electromagnetic field analysis through the time-domain wavelet-Galerkin scheme. The algorithm was verified by the analysis of resonant cavities. The interpolation bases associated with their duals of linear combination of Diracs yield schemes of arbitrary orders of regularity while saving the numerical overhead of field reconstruction process. This is particularly effective for large scale problems with nonlinear media, which will be the subject of further research.

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